Multiport Quantum Teleportation Protocols and Their Performance.

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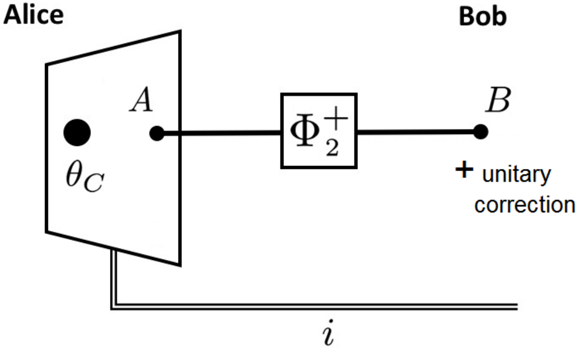
NATIONAL SCIENCE CENTRE POLAND

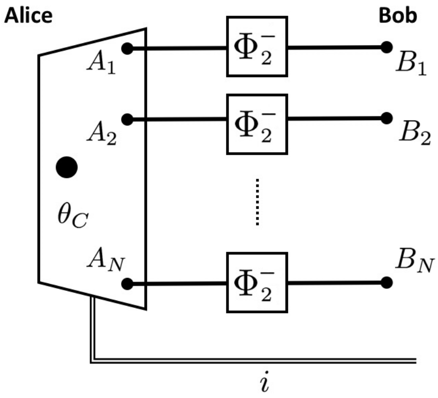
UNIVERSITY OF GDANSK

Quantum Teleportation: *C.H. Bennett et al. PRL* ***70****, 1895-1899 (1993)*



$$\Phi_2^+ = |\psi_2^+\rangle\langle\psi_2^+|, \quad |\psi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Port-based Teleportation (PBT):
S. Ishizaka, T. Hiroshima, PRL **101**, *240501 (2008)*



$$\Phi_2^- = |\psi_2^-\rangle\langle\psi_2^-|, \quad |\psi_2^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Probabilistic and Deterministic PBT

Deterministic Scheme

- We have N measurements $\{\Pi_i\}_{i=1}^N$.
- The state θ_C is always teleported.
- Performance is described by the *entanglement fidelity* F.

Probabilistic Scheme

- We have $N+1$ measurements $\{\Pi_i\}_{i=0}^N$.
- Measurement Π_0 corresponds to failure.
- The state θ_C is teleported perfectly.
- Performance is described by the *probability of success* p.

Applications of PBT

- New architecture for the universal programmable quantum processor
  S. Ishizaka, T. Hiroshima, PRA **79**, *042306 (2009)*
- Efficient attacks for position based cryptography
  S.Beigi, R. König, NJP **13**, *093036 (2011)*
- Fundamental limitations for quantum channels discrimination
  S. Pirandola et al. npj Quantum Information **5**, *50 (2019)*
- Universal simulator of quantum channels
  J. Pereira et al. arXiv:1912.10374
- Aspects of reversing unknown quantum transformations
  M.T. Quintino et al., PRL **123**, *210502 (2019)*
  Talk by Marco Túlio Quintino today!

Mathematical tools in the qubit ($d = 2$) case

S. Ishizaka, T. Hiroshima, PRA **79**, *042306 (2009)*
S. Ishizaka, T. Hiroshima, PRL **101**, *240501 (2008)*

- Correspondence between qubits and spins 1/2

$$|0\rangle, |1\rangle \leftrightarrow |1/2, -1/2\rangle, |1/2, 1/2\rangle$$

- Each qubit is $1/2$ spin $\rightarrow$ basis of $SU(2)$
- In the protocol we have $SU(2)^{\otimes N}$ symmetry $\rightarrow$ representation theory, theory of angular momentum
- Main tolls here: Clebsch-Gordan coefficients + SDP methods

For $d > 2$ new mathematical tools are needed!

Representation Theory of $S(n)$ in a Nutshell

- Let us take permutation group $S(n)$
- For natural number n we define **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$

$$\forall i \ \lambda_i \geq 0, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \quad \sum_{i=1}^{r} \lambda_i = n$$

- Every sequence can be represented graphically $\leftrightarrow$ **Young diagrams**

(4) $(3,1)$ $(2,2)$ $(2,1,1)$ $(1,1,1,1)$

- Irreps denoted by Greek letters, multiplicities by m_μ, dimensions by d_μ etc.

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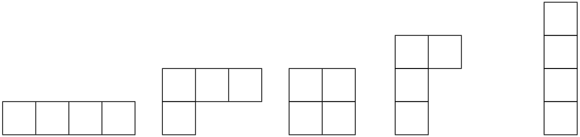

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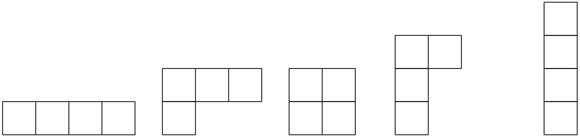

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Schur-Weyl duality

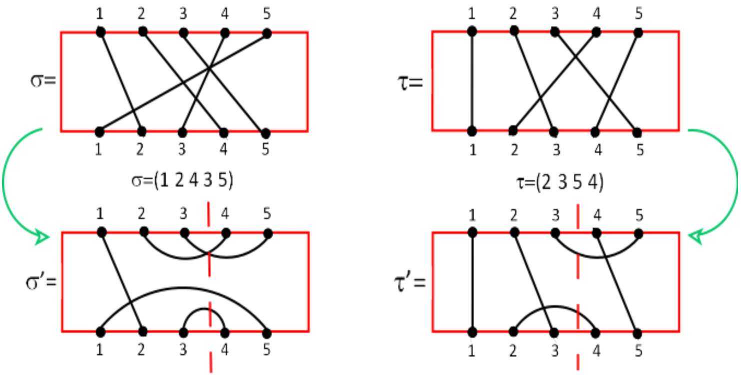
Let us take $|i_1\rangle \otimes \cdots \otimes |i_n\rangle \in (\mathbb{C}^d)^{\otimes n}$.

$$\forall\ \pi \in S(n) \qquad V(\pi)\left(|i_1\rangle \otimes \ldots \otimes |i_n\rangle\right) = |i_{\pi^{-1}(1)}\rangle \otimes \ldots \otimes |i_{\pi^{-1}(n)}\rangle,$$

$$\forall U \in \mathcal{U}(d) \quad U^{\otimes n}\left(|i_1\rangle \otimes \ldots \otimes |i_n\rangle\right) = U|i_1\rangle \otimes \ldots \otimes U|i_n\rangle.$$

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathcal{H}_\lambda^{\mathcal{U}} \otimes \mathcal{H}_\lambda^{\mathcal{S}},$$

Walled Brauer Algebras



Walled Brauer Algebras

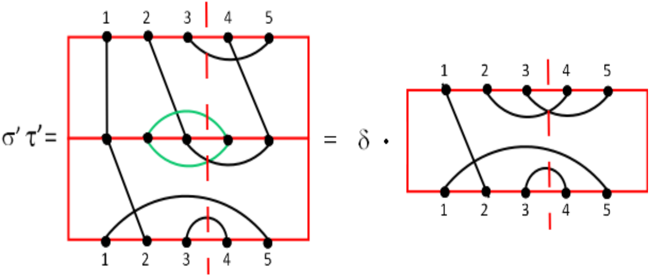
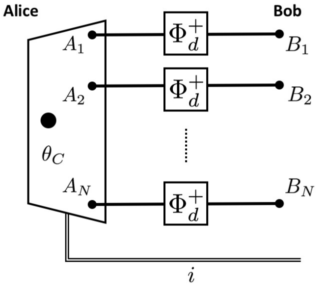


Figure: See for example: R. Brauer *Annals of Mathematics 38 4, 857-872 (1937)* or A. Cox, M. Visscher, S. Doty, and P. Martin, *Journal of Algebra 320, 169-212 (2008).*

Y. Kimura, S. Ramogoolam, Branes, Anti-Branes and Brauer Algebras in Gauge-Gravity duality, JHEP 0711:078 (2007)

Natural Symmetries in PBT



$$\underbrace{U^* \otimes \overbrace{U \otimes \cdots \otimes U}^{N}}_{n}$$

$n = N + 1$,

N – number of ports,

n – N + teleported particle

1. Projection onto maximally entangled state $|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_k |k\rangle \otimes |k\rangle$
2. The state $|\psi_d^+\rangle$ is $U^* \otimes U$ invariant
3. Measurements on Alice's side have *natural $U^* \otimes U \otimes \cdots \otimes U$ symmetry + permutational covariance w.r.t. $S(N)/S(N-1)$*

Natural Symmetries in PBT

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_k |k\rangle \otimes |k\rangle$$

1. Description of the commutant of $U^* \otimes U \otimes \cdots \otimes U$ is needed
2. Complex conjugation translates into *partial transpose*
3. $V(1n)^{t_n} = dP_+ = d|\psi_d^+\rangle\langle\psi_d^+|_{1n} \otimes \mathbf{1} \in \mathcal{A}_d^{t_n}(n)$

Elements describing the performance of PBT belong to $\mathcal{A}_d^{t_n}(n)$.

$$\Pi_i \sim \frac{1}{\sqrt{\rho}} V(1n)^{t_n} \frac{1}{\sqrt{\rho}}, \qquad \rho \sim \sum_{i=1}^{n-1} V(in)^{t_n} \tag{1}$$

Rigorous Definition of Algebra $\mathcal{A}_d^{t_n}(n)$

Definition

For $\mathcal{A}_n(d) = \operatorname{Span}_{\mathbb{C}}\{V(\sigma) : \sigma \in S(n)\}$ we define a new complex algebra

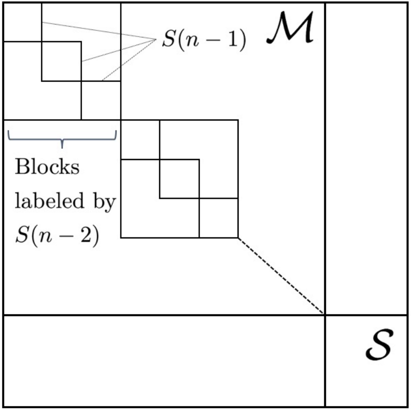
$$\mathcal{A}_d^{t_n}(n) := \operatorname{Span}_{\mathbb{C}}\{V(\sigma)^{t_n} : \sigma \in S(n)\} \subset \operatorname{Hom}((\mathbb{C}^d)^{\otimes n}),$$

where the symbol t_n describes the partial transpose in the last place in the space $\operatorname{Hom}((\mathbb{C}^d)^{\otimes n})$. The elements $V(\sigma)^{t_n} : \sigma \in S(n)$ will be called natural generators of the algebra $\mathcal{A}_d^{t_n}(n)$.

$$V(kn)V(kn) = \mathbf{1} \qquad V(kn)^{t_n}V(kn)^{t_n} = dV(kn)^{t_n}$$

Structure of Algebra $\mathcal{A}_d^{t_n}(n)$

$$\mathcal{A}_d^{t_n}(n) = \mathcal{M} \oplus \mathcal{N}, \qquad \text{support}(\rho) = \mathcal{M}$$



Performance of PBT

- Deterministic case:

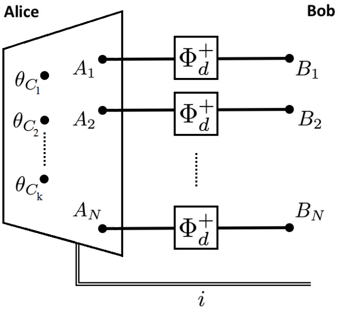
$$F = \frac{1}{d^{N+2}} \sum_{\alpha \vdash N-1} \left(\sum_{\mu=\alpha+\square} \sqrt{d_\mu m_\mu} \right)^2 \sim 1 - \frac{d^2-1}{4N}.$$

- Probabilistic case:

$$p = 1 - \frac{d^2-1}{N+d^2-1}.$$

M. Mozrzymas et al. NJP **20.5**, *053006 (2018)*
M. Studziński et al. Sci. Rep. **7**, *10871 (2017)*
M. Christandl et al., arXiv:1809.10751v1

Teleporting more than one particle



We can use standard PBT protocol:

$$d \to d^k$$

$$F \sim 1 - \frac{d^{2k} - 1}{4N}$$

1. Measurements on Alice's side have:

$$\text{invariance:} \quad \underbrace{U^* \otimes \cdots \otimes U^*}_{k} \otimes U \otimes \cdots \otimes U,$$

$$\text{covariance:} \quad S(N)/S(N-k).$$

2. Extension of $\mathcal{A}_d^{t_n}(n)$ to $\mathcal{A}_d^{t_k}(n)$.

Teleporting more than one particle

- Deterministic case:

$$F = \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left(\sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{d_\mu m_\mu} \right)^2$$

- Probabilistic case:

$$p \leq \frac{N}{d^2 + N - 1} \frac{N-1}{d^2 + N - 2} \cdots \frac{N-k+1}{d^2 + N - k}$$

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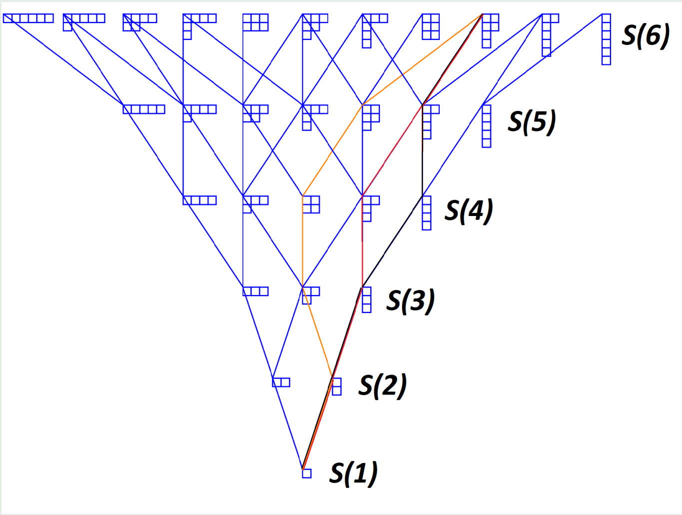

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Summary

What we have:

- Novel mathematical tools for quantum information.
- Full description PBT scheme - deterministic and probabilistic.
- Efficient new multipartite teleportation protocols.

What we have to do:

- Try to optimize the resource state in deterministic version.
- Solve the dual SDP problem for probabilistic case.

Thank you!